

Bi-harmonic superspace for $\mathcal{N} = 4$ $d = 4$ super Yang-Mills

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Abstract

We develop $\mathcal{N} = 4$ $d = 4$ bi-harmonic superspace and use it to derive a novel form for the low-energy effective action in $\mathcal{N} = 4$ super Yang-Mills theory. We solve the $\mathcal{N} = 4$ supergauge constraints in this superspace in terms of analytic superfields. Using these superfields, we construct a simple functional that respects $\mathcal{N} = 4$ supersymmetry and scale invariance. In components, it reproduces all on-shell terms in the four-derivative part of the $\mathcal{N} = 4$ SYM effective action; in particular, the F^4/X^4 and Wess-Zumino terms. The latter comes out in a novel $SO(3) \times SO(3)$ -invariant form.

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Contents

1	Introduction	1
2	$\mathcal{N} = 4$ bi-harmonic superspace	3
2.1	SU(2)×SU(2) harmonics	3
2.2	Tilde-conjugation	4
2.3	Analytic subspaces	5
3	$\mathcal{N} = 4$ SYM in bi-harmonic superspace	5
3.1	Constraints on the $\mathcal{N} = 4$ gauge superfield strength	6
3.2	Harmonic projections and solutions to the constraints	6
3.2.1	Chargeless superfield	7
3.2.2	Charged superfield	9
4	$\mathcal{N} = 4$ SYM effective action	9
4.1	The superfield effective action	9
4.2	Bosonic terms and SO(3) harmonics	10
4.3	The F^4/X^4 term	11
4.4	The WZ term	12
4.5	SU(N) gauge group	13
5	Conclusions	14
A	Covariant derivatives in analytic coordinates	15
A.1	Spinor and harmonic derivatives in the subspace A_1	16
A.2	Spinor and harmonic derivatives in the subspace A_2	16
B	SO(3) harmonic integrals	17

1 Introduction

In spite of many attempts, the problem of formulating the classical action of the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in $\mathcal{N} = 4$ superspace remains unsolved. In our recent paper [1], however, we demonstrated that the $\mathcal{N} = 4$ USp(4) harmonic superspace can be naturally used to describe the low-energy *effective* action of this model. We considered there leading terms in the derivative expansion of the $\mathcal{N} = 4$ SYM low-energy effective action on the Coulomb branch, which are given by the so-called ‘ F^4/X^4 ’ term [2, 3] and the Wess-Zumino (WZ) term [4, 5], and showed that they originate from a simple functional in the $\mathcal{N} = 4$ harmonic superspace with USp(4) harmonic variables. The low-energy effective action in this superspace has a remarkably simple form owing to scale invariance and explicit $\mathcal{N} = 4$ supersymmetry [6]. In this paper, we will describe another superspace where the form of the effective action turns out to be equally simple.

The WZ term in the $\mathcal{N} = 4$ SYM effective action [4, 5] can be written in a manifestly SO(6)~SU(4)-invariant form at the price of sacrificing locality: by writing it as an integral over a five-dimensional manifold that has the four-dimensional Minkowski space as its

boundary. In a local four-dimensional form of the WZ term, only a subgroup of $SU(4)$ can be manifest [7, 8]. In [1], we presented three different forms of the WZ term in the four-dimensional Minkowski space, which are manifestly invariant under $SO(5)$, $SO(4) \times SO(2)$, and $SO(3) \times SO(3)$, respectively. These are nothing but the three maximal non-anomalous subgroups of the $SU(4)$ R-symmetry group. We argued there that the most elegant description of the $\mathcal{N} = 4$ low-energy effective action must be in those superspaces which make these subgroups manifest.

In [1], we matched two of the non-anomalous subgroups, $SO(5)$ and $SO(4) \times SO(2)$, with two known superspace descriptions of the $\mathcal{N} = 4$ SYM effective action. The former corresponds to the $\mathcal{N} = 4$ $USp(4)$ harmonic superspace [6], whereas the latter to the standard $\mathcal{N} = 2$ $SU(2)$ harmonic superspace [9, 10].² However, no superspace was known which would make the $SO(3) \times SO(3)$ subgroup manifest. In this paper, we will fill the gap by introducing $\mathcal{N} = 4$ $d = 4$ bi-harmonic superspace with explicitly realized $SU(2) \times SU(2) \sim SO(3) \times SO(3)$ subgroup of the $SU(4) \sim SO(6)$ R-symmetry group.³ We will demonstrate that the description of the $\mathcal{N} = 4$ SYM low-energy effective action in this superspace is as elegant as in the $USp(4)$ harmonic superspace [1, 6].

The six real scalars X^M , $M = 1, \dots, 6$ in the $\mathcal{N} = 4$ SYM theory transform as a vector of the $SO(6)$ R-symmetry group. So do the normalized scalars $Y^M = X^M/|X|$, which lie on the unit sphere. Splitting them into two triplets, $Y^M = (Y^A, Y^{A'})$, $A, A' = 1, 2, 3$, we find that Y^A and $Y^{A'}$ transform as vectors under different $SO(3)$ subgroups of the $SO(6)$ group. In [1], we showed that the WZ term of the $\mathcal{N} = 4$ SYM effective action can then be written as

$$-\frac{1}{16\pi^2} \varepsilon^{mnpq} \int d^4x g(y) (\varepsilon_{ABC} Y^A \partial_m Y^B \partial_n Y^C) (\varepsilon_{A'B'C'} Y^{A'} \partial_p Y^{B'} \partial_q Y^{C'}), \quad (1.1)$$

where

$$g(y) = \frac{y^4 - 1}{y^2} + \frac{(y^2 + 1)^3}{y^3} \arctan y, \quad y^2 = \frac{Y^A Y^A}{Y^{A'} Y^{A'}}. \quad (1.2)$$

In this form, the WZ term is explicitly invariant under the $SO(3) \times SO(3)$ subgroup of $SO(6)$, whereas the rest of the $SO(6)$ invariance is implicit. As we will see, the WZ term (1.1) follows naturally from a functional in the $\mathcal{N} = 4$ $d = 4$ bi-harmonic superspace.

This paper is organized as follows. In Section 2, we introduce $SU(2) \times SU(2)$ harmonic variables in the standard $\mathcal{N} = 4$ $d = 4$ superspace and classify analytic subspaces in the resulting bi-harmonic superspace. In Section 3, we define analytic (short) superfields in this superspace which solve a part of the $\mathcal{N} = 4$ supergauge constraints. Section 4 is devoted to the construction of superspace actions in terms of one of the analytic superfields. There we show that scale invariance fixes the form of the effective action uniquely, up to a coefficient. In this section, we also study the component structure of the effective action and confirm that it contains the F^4/X^4 and WZ terms. In the Appendices, we have collected some useful formulae for the covariant spinor and harmonic derivatives in the analytic coordinates, as well as for the $SO(3)$ harmonic integrals.

² The $\mathcal{N} = 4$ SYM effective action in the $\mathcal{N} = 2$ harmonic superspace was first constructed in [11], and later rederived through direct perturbative computations in [12, 13, 14, 15].

³We note that similar bi-harmonic superspaces have been used in [16, 17, 18, 19, 20] to describe $d = 2$ supersymmetric sigma models and in [21, 22] to discuss $d = 1$ supersymmetric mechanics with extended supersymmetry.

2 $\mathcal{N} = 4$ bi-harmonic superspace

In this section, we define the basic structures of the $\mathcal{N} = 4$ bi-harmonic superspace.

2.1 $\text{SU}(2) \times \text{SU}(2)$ harmonics

The conventional $\mathcal{N} = 4$ $d = 4$ superspace is described by Minkowski space coordinates x^m and Grassmann coordinates θ_I^α , $\bar{\theta}^{I\dot{\alpha}}$, where $I = 1, 2, 3, 4$ is the $\text{SU}(4)$ index. We are going to constrict a functional in the $\mathcal{N} = 4$ superspace which manifestly respects only the $\text{SU}(2) \times \text{SU}(2) \sim \text{SO}(3) \times \text{SO}(3)$ subgroup of the full $\text{SU}(4) \sim \text{SO}(6)$ R-symmetry group. Therefore, it is convenient to label Grassmann coordinates by the indices of this $\text{SU}(2) \times \text{SU}(2)$ subgroup rather than the full $\text{SU}(4)$ group.

Let $i = 1, 2$ and $a = 1, 2$ be the indices corresponding to the two $\text{SU}(2)$'s. We then represent the $\text{SU}(4)$ index I by the pair (i, a) :

$$I = (i, a) = [(1, 1), (1, 2), (2, 1), (2, 2)], \quad (2.1)$$

and label Grassmann variables as θ_{ia}^α , $\bar{\theta}^{ia\dot{\alpha}}$. Here the bar indicates complex conjugation. The $\text{SU}(2)$ indices can be raised and lowered with the $\text{SU}(2)$ ε -tensors,

$$\overline{(\theta_{ia}^\alpha)} = \bar{\theta}^{ia\dot{\alpha}}, \quad \theta^{ia\alpha} = \varepsilon^{ij} \varepsilon^{ab} \theta_{jb}^\alpha. \quad (2.2)$$

As in [22], we now introduce two sets of harmonic variables, u_i^\pm and v_a^\pm ,

$$u^{+i} u_i^- = v^{+a} v_a^- = 1, \quad u^{+i} u_i^+ = u^{-i} u_i^- = 0, \quad v^{+a} v_a^+ = v^{-a} v_a^- = 0, \quad (2.3)$$

and define the following covariant harmonic derivatives,

$$\begin{aligned} D^{(2,0)} &= u_i^+ \frac{\partial}{\partial u_i^-}, & D^{(-2,0)} &= u_i^- \frac{\partial}{\partial u_i^+}, & S_1 &= [D^{(2,0)}, D^{(-2,0)}] = u_i^+ \frac{\partial}{\partial u_i^+} - u_i^- \frac{\partial}{\partial u_i^-}, \\ D^{(0,2)} &= v_a^+ \frac{\partial}{\partial v_a^-}, & D^{(0,-2)} &= v_a^- \frac{\partial}{\partial v_a^+}, & S_2 &= [D^{(0,2)}, D^{(0,-2)}] = v_a^+ \frac{\partial}{\partial v_a^+} - v_a^- \frac{\partial}{\partial v_a^-}. \end{aligned} \quad (2.4)$$

The operators S_1 and S_2 define $\text{U}(1)$ subgroups of the two $\text{SU}(2)$'s and measure $\text{U}(1)$ charges of other operators: $[S_1, D^{(s_1, s_2)}] = s_1 D^{(s_1, s_2)}$, $[S_2, D^{(s_1, s_2)}] = s_2 D^{(s_1, s_2)}$. Accordingly, we define the following bi-harmonic projections of the Grassmann variables,

$$\begin{aligned} \theta_\alpha^{(1,1)} &= u_i^+ v_a^+ \theta_\alpha^{ia}, & \theta_\alpha^{(1,-1)} &= u_i^+ v_a^- \theta_\alpha^{ia}, & \theta_\alpha^{(-1,1)} &= u_i^- v_a^+ \theta_\alpha^{ia}, & \theta_\alpha^{(-1,-1)} &= u_i^- v_a^- \theta_\alpha^{ia}, \\ \bar{\theta}_{\dot{\alpha}}^{(1,1)} &= u_i^+ v_a^+ \bar{\theta}_{\dot{\alpha}}^{ia}, & \bar{\theta}_{\dot{\alpha}}^{(1,-1)} &= u_i^+ v_a^- \bar{\theta}_{\dot{\alpha}}^{ia}, & \bar{\theta}_{\dot{\alpha}}^{(-1,1)} &= u_i^- v_a^+ \bar{\theta}_{\dot{\alpha}}^{ia}, & \bar{\theta}_{\dot{\alpha}}^{(-1,-1)} &= u_i^- v_a^- \bar{\theta}_{\dot{\alpha}}^{ia}, \end{aligned} \quad (2.5)$$

where superscripts indicate the $\text{U}(1)$ charges. To make the subsequent expressions more compact, however, we introduce a single (bold) index to represent the pairs of $\text{U}(1)$ charges. Namely, we define

$$\begin{aligned} \theta_\alpha^{\mathbf{1}} &\equiv \theta_\alpha^{(1,1)}, & \theta_\alpha^{\mathbf{2}} &\equiv \theta_\alpha^{(1,-1)}, & \theta_\alpha^{\mathbf{3}} &\equiv \theta_\alpha^{(-1,1)}, & \theta_\alpha^{\mathbf{4}} &\equiv \theta_\alpha^{(-1,-1)}, \\ \bar{\theta}_{\dot{\alpha}}^{\mathbf{1}} &\equiv \bar{\theta}_{\dot{\alpha}}^{(-1,-1)}, & \bar{\theta}_{\dot{\alpha}}^{\mathbf{2}} &\equiv \bar{\theta}_{\dot{\alpha}}^{(-1,1)}, & \bar{\theta}_{\dot{\alpha}}^{\mathbf{3}} &\equiv \bar{\theta}_{\dot{\alpha}}^{(1,-1)}, & \bar{\theta}_{\dot{\alpha}}^{\mathbf{4}} &\equiv \bar{\theta}_{\dot{\alpha}}^{(1,1)}. \end{aligned} \quad (2.6)$$

We emphasize that these θ 's have definite U(1) charges and are linear combinations of the original θ 's with SU(4) indices.

Going through the same steps for the standard covariant spinor derivatives $D_\alpha^I, \bar{D}_{I\dot{\alpha}}$,

$$D_\alpha^I = \frac{\partial}{\partial \theta_I^\alpha} + i\bar{\theta}^{I\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \quad \bar{D}_{I\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{I\dot{\alpha}}} - i\theta_I^\alpha\partial_{\alpha\dot{\alpha}}, \quad \{D_\alpha^I, \bar{D}_{J\dot{\alpha}}\} = -2i\delta_J^I\partial_{\alpha\dot{\alpha}}, \quad (2.7)$$

we define their bi-harmonic projections as

$$\begin{aligned} D_\alpha^1 &= +\frac{\partial}{\partial \theta^{1\alpha}} + i\bar{\theta}^{1\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^1 &= -\frac{\partial}{\partial \bar{\theta}^{1\dot{\alpha}}} - i\theta^{1\alpha}\partial_{\alpha\dot{\alpha}} \\ D_\alpha^2 &= -\frac{\partial}{\partial \theta^{2\alpha}} + i\bar{\theta}^{2\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^2 &= +\frac{\partial}{\partial \bar{\theta}^{2\dot{\alpha}}} - i\theta^{2\alpha}\partial_{\alpha\dot{\alpha}} \\ D_\alpha^3 &= -\frac{\partial}{\partial \theta^{3\alpha}} + i\bar{\theta}^{3\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^3 &= +\frac{\partial}{\partial \bar{\theta}^{3\dot{\alpha}}} - i\theta^{3\alpha}\partial_{\alpha\dot{\alpha}} \\ D_\alpha^4 &= +\frac{\partial}{\partial \theta^{4\alpha}} + i\bar{\theta}^{4\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, & \bar{D}_{\dot{\alpha}}^4 &= -\frac{\partial}{\partial \bar{\theta}^{4\dot{\alpha}}} - i\theta^{4\alpha}\partial_{\alpha\dot{\alpha}}. \end{aligned} \quad (2.8)$$

The non-trivial anticommutation relations among them are given by

$$\{D_\alpha^1, \bar{D}_{\dot{\alpha}}^1\} = \{D_\alpha^4, \bar{D}_{\dot{\alpha}}^4\} = -2i\partial_{\alpha\dot{\alpha}}, \quad \{D_\alpha^2, \bar{D}_{\dot{\alpha}}^2\} = \{D_\alpha^3, \bar{D}_{\dot{\alpha}}^3\} = 2i\partial_{\alpha\dot{\alpha}}. \quad (2.9)$$

2.2 Tilde-conjugation

Proper definition of conjugation in harmonic superspaces is essential for defining ‘real’ objects. In the case at hand, complex conjugation is uniquely specified by its standard action on the SU(2) harmonics,

$$\overline{u^+} = u_i^-, \quad \overline{u_i^-} = -u^+, \quad \overline{v^+} = v_a^-, \quad \overline{v_a^-} = -v^+. \quad (2.10)$$

However, complex conjugation turns out to be inadequate for use in analytic subspaces that we will introduce next. Instead, we will need the so-called tilde-conjugation “ \sim ” defined as a combination of the complex conjugation “ $\overline{}$ ” and a special involution “ \star ”.

In the standard $\mathcal{N} = 2$ harmonic superspace [9, 10], the \star -involution is defined by

$$(u^+)^{\star} = u^-, \quad (u_i^-)^{\star} = u_i^+, \quad (u^-)^{\star} = -u^+, \quad (u_i^+)^{\star} = -u_i^-. \quad (2.11)$$

It acts only on the harmonic variables and squares to -1 on them. In the bi-harmonic superspace, however, we have two independent sets of harmonics variables, u_i^\pm and v_a^\pm , and there are several ways in which the \star -involution can be defined. We will define it to act on u_i^\pm by the rule (2.11) while leaving the harmonics v_a^\pm inert,

$$(v_a^\pm)^{\star} = v_a^\pm, \quad (v^{\pm a})^{\star} = v^{\pm a}. \quad (2.12)$$

Combining the complex conjugation (2.10) with the \star -involution (2.11, 2.12), we obtain the tilde-conjugation that acts on the harmonics and Grassmann variables as follows,

$$\begin{aligned} \widetilde{u_i^\pm} &= u_i^\pm, & \widetilde{u^{\pm i}} &= -u_i^\pm, \\ \widetilde{v^+} &= v_a^-, & \widetilde{v_a^+} &= -v^+, & \widetilde{v^-} &= v_a^+, & \widetilde{v_a^-} &= v^+, \end{aligned} \quad (2.13)$$

$$\begin{aligned} \widetilde{\theta_\alpha^1} &= -\bar{\theta}_\alpha^3, & \widetilde{\theta_\alpha^2} &= \bar{\theta}_\alpha^4, & \widetilde{\theta_\alpha^3} &= -\bar{\theta}_\alpha^1, & \widetilde{\theta_\alpha^4} &= \bar{\theta}_\alpha^2, \\ \widetilde{\bar{\theta}_\alpha^1} &= \theta_\alpha^3, & \widetilde{\bar{\theta}_\alpha^2} &= -\theta_\alpha^4, & \widetilde{\bar{\theta}_\alpha^3} &= \theta_\alpha^1, & \widetilde{\bar{\theta}_\alpha^4} &= -\theta_\alpha^2. \end{aligned} \quad (2.14)$$

As we will see later, this is exactly the conjugation that will allow us to introduce real actions in the $\mathcal{N} = 4$ bi-harmonic superspace.

2.3 Analytic subspaces

By definition, an analytic subspace in the full $\mathcal{N} = 4$ superspace must (i) depend on half of Grassmann variables of the full superspace, (ii) be closed under $\mathcal{N} = 4$ supersymmetry, (iii) contain an equal number of θ and $\bar{\theta}$ variables. (For comparison, chiral subspaces depend on θ variables only.) To construct such subspaces, we pass from standard bosonic coordinates x^m to the analytic ones,

$$x_A^m = x^m + a_1(i\theta^1\sigma^m\bar{\theta}^1) + a_2(i\theta^2\sigma^m\bar{\theta}^2) + a_3(i\theta^3\sigma^m\bar{\theta}^3) + a_4(i\theta^4\sigma^m\bar{\theta}^4), \quad (2.15)$$

where $a_k = \pm 1$. Choosing $+1$ or -1 for each a_k fixes the corresponding analytic subspace. In fact, there are six such analytic subspaces, corresponding to six different ways of choosing two out of four Grassmann variables:

	coordinates	(a_1, a_2, a_3, a_4)	short derivatives
A_1	$\{x_{A_1}^m, \theta_\alpha^1, \bar{\theta}_\alpha^2, \bar{\theta}_\alpha^3, \theta_\alpha^4, u, v\}$	$(+, +, +, +)$	$\bar{D}_\alpha^1, D_\alpha^2, D_\alpha^3, \bar{D}_\alpha^4$
\bar{A}_1	$\{x_{\bar{A}_1}^m, \bar{\theta}_\alpha^1, \theta_\alpha^2, \theta_\alpha^3, \bar{\theta}_\alpha^4, u, v\}$	$(-, -, -, -)$	$D_\alpha^1, \bar{D}_\alpha^2, \bar{D}_\alpha^3, D_\alpha^4$
A_2	$\{x_{A_2}^m, \theta_\alpha^1, \bar{\theta}_\alpha^2, \bar{\theta}_\alpha^3, \bar{\theta}_\alpha^4, u, v\}$	$(+, -, +, -)$	$\bar{D}_\alpha^1, \bar{D}_\alpha^2, D_\alpha^3, D_\alpha^4$
\bar{A}_2	$\{x_{\bar{A}_2}^m, \bar{\theta}_\alpha^1, \bar{\theta}_\alpha^2, \theta_\alpha^3, \theta_\alpha^4, u, v\}$	$(-, +, -, +)$	$D_\alpha^1, D_\alpha^2, \bar{D}_\alpha^3, \bar{D}_\alpha^4$
A_3	$\{x_{A_3}^m, \theta_\alpha^1, \bar{\theta}_\alpha^2, \theta_\alpha^3, \bar{\theta}_\alpha^4, u, v\}$	$(+, +, -, -)$	$\bar{D}_\alpha^1, D_\alpha^2, \bar{D}_\alpha^3, D_\alpha^4$
\bar{A}_3	$\{x_{\bar{A}_3}^m, \bar{\theta}_\alpha^1, \theta_\alpha^2, \bar{\theta}_\alpha^3, \theta_\alpha^4, u, v\}$	$(-, -, +, +)$	$D_\alpha^1, \bar{D}_\alpha^2, D_\alpha^3, \bar{D}_\alpha^4$

(2.16)

In the last column, we listed covariant spinor derivatives that become ‘short’ in the corresponding coordinates. They differentiate along those Grassmann directions which are *orthogonal* to the corresponding analytic subspace.

Under the tilde-conjugation (2.13, 2.14), the analytic subspaces A_1 , \bar{A}_1 , A_2 and \bar{A}_2 are *real*, whereas A_3 and \bar{A}_3 transform into each other.⁴ In what follows, we will explicitly consider only the subspaces A_1 and A_2 . The expressions for the covariant spinor (2.8) and harmonic (2.4) derivatives in these subspaces are given in Appendix A.

The remaining analytic subspaces could be treated similarly, but this would not produce any qualitatively new results.

3 $\mathcal{N} = 4$ SYM in bi-harmonic superspace

In this section, we start with the standard $\mathcal{N} = 4$ supergauge field strength and use it to define six different harmonic superfields, each of which *independently* can be used to describe the on-shell $\mathcal{N} = 4$ SYM multiplet.

⁴The subspaces A_3 and \bar{A}_3 would be real under *modified* tilde-conjugation where the behavior of the harmonics u and v with respect to the \star -involution (2.11, 2.12) is reversed.

3.1 Constraints on the $\mathcal{N} = 4$ gauge superfield strength

In the conventional $\mathcal{N} = 4$ superspace, the $\mathcal{N} = 4$ superfield strength is described by an antisymmetric $\text{SU}(4)$ tensor, $W^{IJ} = -W^{JI}$, subject to the following constraints [23, 24],

$$\overline{W^{IJ}} \equiv \bar{W}_{IJ} = \frac{1}{2} \varepsilon_{IJKL} W^{KL}, \quad (3.1)$$

$$D_\alpha^I W^{JK} + D_\alpha^J W^{IK} = 0, \quad (3.2)$$

$$\bar{D}_{I\dot{\alpha}} W^{JK} = \frac{1}{3} (\delta_I^J \bar{D}_{L\dot{\alpha}} W^{LK} - \delta_I^K \bar{D}_{L\dot{\alpha}} W^{LJ}). \quad (3.3)$$

Writing the $\text{SU}(4)$ indices as pairs of $\text{SU}(2)$ ones, as in (2.1), we find

$$W^{IJ} \equiv W^{ia,jb} = \varepsilon^{ij} W^{ab} + \varepsilon^{ab} W^{ij}, \quad (3.4)$$

so that the superfield strength W^{IJ} becomes represented by a pair of *symmetric* $\text{SU}(2)$ tensors: $W^{ab} = W^{ba}$ and $W^{ij} = W^{ji}$. The constraints (3.1)–(3.3) can be readily rewritten for these tensors. In particular, using the identity

$$\varepsilon_{IJKL} \equiv \varepsilon_{ia,jb,kc,ld} = \varepsilon_{il} \varepsilon_{jk} \varepsilon_{ab} \varepsilon_{cd} - \varepsilon_{ij} \varepsilon_{kl} \varepsilon_{ad} \varepsilon_{bc}, \quad (3.5)$$

we find that (3.1) corresponds to the following reality properties of the $\text{SU}(2)$ tensors,

$$\overline{W^{ij}} \equiv \bar{W}_{ij} = W_{ij}, \quad \overline{W^{ab}} \equiv \bar{W}_{ab} = -W_{ab}. \quad (3.6)$$

The constraint (3.2) is equivalent to the following set of constraints,

$$D_\alpha^{a(i} W^{jk)} = 0, \quad D_\alpha^{i(a} W^{bc)} = 0, \quad D_\alpha^{ka} W_k^i + D_\alpha^{ic} W_c^a = 0. \quad (3.7)$$

The constraint (3.3) leads to equations conjugate to these,⁵

$$\bar{D}_{\dot{\alpha}}^{a(i} W^{jk)} = 0, \quad \bar{D}_{\dot{\alpha}}^{i(a} W^{bc)} = 0, \quad \bar{D}_{\dot{\alpha}}^{ka} W_k^i - \bar{D}_{\dot{\alpha}}^{ic} W_c^a = 0. \quad (3.8)$$

We conclude that equations (3.6), (3.7) and (3.8) are equivalent to the $\mathcal{N} = 4$ supergauge constraints (3.1), (3.2) and (3.3).

3.2 Harmonic projections and solutions to the constraints

Now we introduce the harmonic projections for the superfields W^{ij} and W^{ab} ,

$$W = u_i^+ u_j^- W^{ij} - v_a^+ v_b^- W^{ab}, \quad W' = u_i^+ u_j^- W^{ij} + v_a^+ v_b^- W^{ab}, \quad (3.9)$$

$$W^{(2,0)} = u_i^+ u_j^+ W^{ij}, \quad W^{(-2,0)} = u_i^- u_j^- W^{ij}, \quad (3.10)$$

$$W^{(0,2)} = v_a^+ v_b^+ W^{ab}, \quad W^{(0,-2)} = v_a^- v_b^- W^{ab}. \quad (3.11)$$

According to the conjugation rules (2.13) and (3.6), these harmonic projections have the following reality properties,

$$\widetilde{W} = W, \quad \widetilde{W'} = W', \quad \widetilde{W^{(\pm 2,0)}} = W^{(\pm 2,0)}, \quad \widetilde{W^{(0,\pm 2)}} = -W^{(0,\mp 2)}. \quad (3.12)$$

⁵ It is straightforward to show that (3.1) and (3.2) together imply (3.3).

Contracting the constraints (3.7) and (3.8) with various combinations of harmonic variables, we find the following first-order differential constraints,

$$\{\bar{D}_\alpha^1, D_\alpha^2, D_\alpha^3, \bar{D}_\alpha^4\}W = 0, \quad (3.13)$$

$$\{D_\alpha^1, \bar{D}_\alpha^2, \bar{D}_\alpha^3, D_\alpha^4\}W' = 0, \quad (3.14)$$

$$\{\bar{D}_\alpha^1, \bar{D}_\alpha^2, D_\alpha^3, D_\alpha^4\}W^{(2,0)} = 0, \quad (3.15)$$

$$\{D_\alpha^1, D_\alpha^2, \bar{D}_\alpha^3, \bar{D}_\alpha^4\}W^{(-2,0)} = 0, \quad (3.16)$$

$$\{\bar{D}_\alpha^1, D_\alpha^2, \bar{D}_\alpha^3, D_\alpha^4\}W^{(0,2)} = 0, \quad (3.17)$$

$$\{D_\alpha^1, \bar{D}_\alpha^2, D_\alpha^3, \bar{D}_\alpha^4\}W^{(0,-2)} = 0, \quad (3.18)$$

as well as additional first-order differential constraints that *mix* different harmonic projections. However, the mixing can be removed at the price of generating the following *second-order* differential constraints,

$$\{(D^1)^2, (\bar{D}^2)^2, (\bar{D}^3)^2, (D^4)^2, (D^1 D^4), (\bar{D}^2 \bar{D}^3)\}W = 0, \quad (3.19)$$

$$\{(\bar{D}^1)^2, (D^2)^2, (D^3)^2, (\bar{D}^4)^2, (\bar{D}^1 \bar{D}^4), (D^2 D^3)\}W' = 0, \quad (3.20)$$

$$\{(D^1)^2, (D^2)^2, (\bar{D}^3)^2, (\bar{D}^4)^2, (D^1 D^2), (\bar{D}^3 \bar{D}^4)\}W^{(2,0)} = 0, \quad (3.21)$$

$$\{(\bar{D}^1)^2, (\bar{D}^2)^2, (D^3)^2, (D^4)^2, (\bar{D}^1 \bar{D}^2), (D^3 D^4)\}W^{(-2,0)} = 0, \quad (3.22)$$

$$\{(D^1)^2, (\bar{D}^2)^2, (D^3)^2, (\bar{D}^4)^2, (D^1 D^3), (\bar{D}^2 \bar{D}^4)\}W^{(0,2)} = 0, \quad (3.23)$$

$$\{(\bar{D}^1)^2, (D^2)^2, (\bar{D}^3)^2, (D^4)^2, (\bar{D}^1 \bar{D}^3), (D^2 D^4)\}W^{(0,-2)} = 0. \quad (3.24)$$

Finally, there are many differential relations for the superfield strengths involving covariant *harmonic* derivatives, which follow from the definitions (3.9)–(3.11). The basic constraints in this set are

$$D^{(2,0)} D^{(2,0)} W = D^{(0,2)} D^{(0,2)} W = D^{(2,0)} D^{(0,2)} W = 0, \quad (3.25)$$

$$D^{(2,0)} D^{(2,0)} W' = D^{(0,2)} D^{(0,2)} W' = D^{(2,0)} D^{(0,2)} W' = 0, \quad (3.26)$$

$$D^{(2,0)} W^{(2,0)} = D^{(0,2)} W^{(2,0)} = D^{(0,-2)} W^{(2,0)} = 0, \quad (3.27)$$

$$D^{(-2,0)} W^{(-2,0)} = D^{(0,2)} W^{(-2,0)} = D^{(0,-2)} W^{(-2,0)} = 0, \quad (3.28)$$

$$D^{(0,2)} W^{(0,2)} = D^{(2,0)} W^{(0,2)} = D^{(-2,0)} W^{(0,2)} = 0, \quad (3.29)$$

$$D^{(0,-2)} W^{(0,-2)} = D^{(2,0)} W^{(0,-2)} = D^{(-2,0)} W^{(0,-2)} = 0. \quad (3.30)$$

Note that the constraints for the chargeless superfields W and W' are *quadratic* in the harmonic derivatives, whereas those for the charged superfields $W^{(\pm 2,0)}$ and $W^{(0,\pm 2)}$ are *linear* in these derivatives.

Our claim now is that equations (3.12)–(3.30) form the complete set of constraints that eliminate all the auxiliary fields in the component expansions of W , W' , $W^{(\pm 2,0)}$, $W^{(0,\pm 2)}$, reducing each of them to the *on-shell* $\mathcal{N} = 4$ supergauge multiplet. We will demonstrate this next by giving explicit solutions of these constraints for the two inequivalent cases: the chargeless superfield W and the charged superfield $W^{(2,0)}$. The other four cases yield qualitatively similar results.

3.2.1 Chargeless superfield

The chargeless superfield W obeys the constraints (3.12), (3.13), (3.19), and (3.25). The constraints (3.13) are nothing but the analyticity conditions. They are solved by passing

to the coordinates of the analytic subspace A_1 given in (2.16),

$$W = W(x_{A_1}^m, \theta_\alpha^1, \bar{\theta}_\alpha^2, \bar{\theta}_\alpha^3, \theta_\alpha^4, u, v). \quad (3.31)$$

Although this superfield depends on half of the Grassmann variables of the $\mathcal{N} = 4$ superspace, its component field decomposition is still quite long. However, it becomes shorter upon taking into account the linearity conditions (3.19). To the remaining components we have to apply the constraints with the covariant harmonic derivatives (3.25). Note that after passing to the analytic coordinates these constraints become dynamical because the covariant harmonic derivatives (A.2) involve space-time derivatives. Taking into account all these equations, we obtain the following component field decomposition for W ,

$$\begin{aligned} W = & \omega + u_i^+ u_j^- \phi^{ij} + v_a^+ v_b^- i\varphi^{ab} \\ & + \theta^{1\alpha} \psi_\alpha^{ia} u_i^- v_a^- - \theta^{4\alpha} \psi_\alpha^{ia} u_i^+ v_a^+ + \bar{\theta}_\alpha^2 \bar{\psi}^{\dot{\alpha}ia} u_i^+ v_a^- - \bar{\theta}_\alpha^3 \bar{\psi}^{\dot{\alpha}ia} u_i^- v_a^+ \\ & + \frac{1}{\sqrt{2}} (\theta_\alpha^1 \theta_\beta^4 \sigma^{m\alpha}{}_{\dot{\alpha}} \sigma^{n\beta\dot{\alpha}} + \bar{\theta}_\alpha^3 \bar{\theta}_\beta^2 \sigma^{m\dot{\alpha}}{}_\alpha \sigma^{n\alpha\dot{\beta}}) F_{mn} \\ & + 2\theta^{1\alpha} \bar{\theta}^{2\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^{ab} v_a^- v_b^- + 2\theta^{4\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^{ab} v_a^+ v_b^+ \\ & - 2i\theta^{4\alpha} \bar{\theta}^{2\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi^{ij} u_i^+ u_j^+ - 2i\theta^{1\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi^{ij} u_i^- u_j^- \\ & + 2i\theta^{4\alpha} \theta^{1\beta} \bar{\theta}^{3\dot{\beta}} \partial_{\beta\dot{\beta}} \psi_\alpha^{ia} u_i^- v_a^+ - 2i\theta^{1\alpha} \theta^{4\beta} \bar{\theta}^{2\dot{\beta}} \partial_{\beta\dot{\beta}} \psi_\alpha^{ia} u_i^+ v_a^- \\ & + 2i\bar{\theta}^{3\dot{\alpha}} \theta^{4\beta} \bar{\theta}^{2\dot{\beta}} \partial_{\beta\dot{\beta}} \bar{\psi}_\alpha^{\dot{a}i} u_i^+ v_a^+ - 2i\bar{\theta}^{2\dot{\alpha}} \theta^{1\beta} \bar{\theta}^{3\dot{\beta}} \partial_{\beta\dot{\beta}} \bar{\psi}_\alpha^{\dot{a}i} u_i^- v_a^- \\ & + 4\theta^{1\alpha} \theta^{4\beta} \bar{\theta}^{3\dot{\alpha}} \bar{\theta}^{2\dot{\beta}} \partial_{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}} (u_i^+ u_j^- \phi^{ij} - v_a^+ v_b^- i\varphi^{ab}). \end{aligned} \quad (3.32)$$

Here ω is a *constant*, $\phi^{ij} = \phi^{(ij)}$ and $\varphi^{ab} = \varphi^{(ab)}$ are two triplets of scalar fields, ψ_α^{ia} are four Weyl spinors and F_{mn} is the Maxwell field strength. These fields obey their classical equations of motion,

$$\square \phi^{ij} = \square \varphi^{ab} = 0, \quad \partial^{\alpha\dot{\alpha}} \psi_\alpha^{ia} = 0, \quad \partial^m F_{mn} = 0. \quad (3.33)$$

No auxiliary field components remain in W as they all have vanished under the constraints (3.13), (3.19) and (3.25).

Let us comment on the constant ω in (3.32). This constant would never have arisen if we started with the component form of W^{IJ} that solves (3.1)–(3.3), constructed W using (3.9) and made the transformation to analytic coordinates A_1 . However, we instead considered W to be *defined* by the constraints (3.12), (3.13), (3.19), and (3.25). These constraints were, indeed, sufficient to properly restrict the component degrees of freedom, except for the residual appearance of this extra constant parameter.

We will set ω to zero by insisting that W transforms linearly under scale transformations, with a constant parameter λ ,

$$\delta W = \lambda W \quad \Rightarrow \quad \omega = 0. \quad (3.34)$$

This requirement is particularly natural for the purposes of the next section, where we will construct the superconformal effective action in the $\mathcal{N} = 4$ supergauge theory.

3.2.2 Charged superfield

Now let us briefly consider the charged superfield $W^{(2,0)}$ subject to the differential constraints (3.15), (3.21), (3.27) and the reality condition (3.12). The analyticity constraints (3.15) are solved by passing to the analytic coordinates A_2 given in (2.16),

$$W^{(2,0)} = W^{(2,0)}(x_{A_2}^m, \theta_\alpha^1, \theta_\alpha^2, \bar{\theta}_{\dot{\alpha}}^3, \bar{\theta}_{\dot{\alpha}}^4, u, v). \quad (3.35)$$

Using the linearity conditions (3.21) and the ‘harmonic shortness’ constraints (3.27), we obtain

$$\begin{aligned} W^{(2,0)} = & \phi^{ij} u_i^+ u_j^+ \\ & + \theta^{1\alpha} \psi_\alpha^{ia} u_i^+ v_a^- - \theta^{2\alpha} \psi_\alpha^{ia} u_i^+ v_a^+ + \bar{\theta}_{\dot{\alpha}}^4 \bar{\psi}^{ia\dot{\alpha}} u_i^+ v_a^- - \bar{\theta}_{\dot{\alpha}}^3 \bar{\psi}^{ia\dot{\alpha}} u_i^+ v_a^+ \\ & + \frac{1}{\sqrt{2}} (\theta_\alpha^1 \theta_\beta^2 \sigma^{m\alpha}_{\dot{\alpha}} \sigma^{n\beta\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}}^3 \bar{\theta}_{\dot{\beta}}^4 \sigma^{m\dot{\alpha}}_{\alpha} \sigma^{n\alpha\dot{\beta}}) F_{mn} \\ & + 2\theta^{2\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^{ab} v_a^+ v_b^- + 2i\theta^{2\alpha} \bar{\theta}^{4\dot{\alpha}} \partial_{\alpha\dot{\alpha}} (\phi^{ij} u_i^+ u_j^- + i\varphi^{ab} v_a^+ v_b^-) \\ & + 2\theta^{1\alpha} \bar{\theta}^{4\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^{ab} v_a^- v_b^- + 2i\theta^{1\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} (-\phi^{ij} u_i^+ u_j^- + i\varphi^{ab} v_a^+ v_b^-) \\ & + 2i\theta^{1\alpha} \theta^{2\beta} \bar{\theta}^{4\dot{\beta}} \partial_{\beta\dot{\beta}} \psi_\alpha^{ia} u_i^- v_a^- + 2i\theta^{2\alpha} \theta^{1\beta} \bar{\theta}^{3\dot{\beta}} \partial_{\beta\dot{\beta}} \psi_\alpha^{ia} u_i^- v_a^+ \\ & + 2i\bar{\theta}^{3\dot{\alpha}} \theta^{2\beta} \bar{\theta}^{4\dot{\beta}} \partial_{\beta\dot{\beta}} \bar{\psi}_{\dot{\alpha}}^{ia} u_i^- v_a^+ + 2i\bar{\theta}^{4\dot{\alpha}} \theta^{1\beta} \bar{\theta}^{3\dot{\beta}} \partial_{\beta\dot{\beta}} \bar{\psi}_{\dot{\alpha}}^{ia} u_i^- v_a^- \\ & - 4\theta^{1\alpha} \theta^{2\beta} \bar{\theta}^{3\dot{\alpha}} \bar{\theta}^{4\dot{\beta}} \partial_{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}} \phi^{ij} u_i^- u_j^-. \end{aligned} \quad (3.36)$$

All auxiliary field components have vanished, whereas the physical components are required to satisfy the free equations of motion (3.33). In this case, unlike (3.32), no extra constant parameter appears in the solution. Note, however, that $W^{(2,0)}$ involves only half of the scalars, ϕ^{ij} , undifferentiated; the remaining scalars, φ^{ab} , appear only with the derivatives acting on them. This limits the range of applications of the charged superfield, and makes W the preferred choice for the description of the $\mathcal{N} = 4$ SYM multiplet.

4 $\mathcal{N} = 4$ SYM effective action

In this section, we find the bi-harmonic superspace form of the $\mathcal{N} = 4$ SYM effective action on the Coulomb branch, and confirm that it correctly reproduces the F^4/X^4 and WZ terms.

4.1 The superfield effective action

The simplest $\mathcal{N} = 4$ superspace action for the superfield W is given by

$$\Gamma = \int d\zeta du dv H(W), \quad (4.1)$$

where $H(W)$ is some function of W without derivatives. The integration goes over the analytic superspace A_1 given in (2.16) with the analytic measure defined so that

$$d\zeta = d^4 x d^8 \theta, \quad \int d^8 \theta (\theta^1)^2 (\theta^4)^2 (\bar{\theta}^2)^2 (\bar{\theta}^3)^2 = 1. \quad (4.2)$$

We use the standard definition for the harmonic integrals [9, 10],

$$\int du \, 1 = 1, \quad \int du (\text{non-singlet SU(2) irreducible representation}) = 0, \quad (4.3)$$

and similarly for the integration over dv . We point out that the function $H(W)$ must have zero $U(1)$ charges since the integration measure $d\zeta$ in the analytic superspace A_1 is chargeless.

Note that the integration measure (4.2) yields eight Grassmann derivatives, or, equivalently, four space-time ones. Therefore, we expect that the action (4.1) with a particular H describes the four-derivative term in the $\mathcal{N} = 4$ low-energy effective action, and that this term is the leading one in the derivative expansion. We will now determine the appropriate function H by requiring scale invariance of the action (4.1), in exactly the same way as we did in [1].

As the measure $d\zeta$ is dimensionless, the function $H(W)$ must also be dimensionless. Recalling that W has mass dimension one, we are forced to introduce a parameter Λ such that W/Λ is dimensionless, and take $H = H(W/\Lambda)$. However, the dependence on Λ should disappear after the integration over Grassmann variables. This uniquely fixes this function in the form

$$H = c \ln \frac{W}{\Lambda}, \quad (4.4)$$

with some coefficient c . Indeed, rescaling W then shifts the integrand in (4.1) by a constant which gives zero under the integral over the Grassmann variables.

In the following, we will demonstrate that the action (4.1) with the function H given by (4.4) does contain the known bosonic terms in the $\mathcal{N} = 4$ SYM effective action. The parameter Λ drops out in the final results, and to simplify the following expressions we formally set $\Lambda = 1$ from now on.

4.2 Bosonic terms and SO(3) harmonics

In the bosonic part of W in (3.32), that is after setting $\psi_\alpha^{ia} = \bar{\psi}_\alpha^{ia} = 0$, only the following harmonic monomials appear: $u_i^+ u_j^+$, $u_i^- u_j^-$, $u_i^+ u_j^-$, and $v_a^+ v_b^+$, $v_a^- v_b^-$, $v_a^+ v_b^-$. For computational reasons, it is convenient to rewrite these $SU(2)$ monomials in terms of $SO(3)$ harmonics $U_{A'}^1$, $U_{A'}^2$, $U_{A'}^3$ and V_A^1 , V_A^2 , V_A^3 ,

$$\begin{aligned} V_A^1 &= i\gamma_A^{ab} u_a^+ u_b^-, & V_A^2 &= \frac{1}{2}\gamma_A^{ab}(u_a^+ u_b^+ + u_a^- u_b^-), & V_A^3 &= \frac{i}{2}\gamma_A^{ab}(u_a^+ u_b^+ - u_a^- u_b^-), \\ U_{A'}^1 &= i\gamma_{A'}^{ij} u_i^+ u_j^-, & U_{A'}^2 &= \frac{1}{2}\gamma_{A'}^{ij}(u_i^+ u_j^+ + u_i^- u_j^-), & U_{A'}^3 &= \frac{i}{2}\gamma_{A'}^{ij}(u_i^+ u_j^+ - u_i^- u_j^-), \end{aligned} \quad (4.5)$$

where γ_{ab}^A , $\gamma_{ij}^{A'}$ are two copies of $SO(3)$ gamma-matrices,

$$\gamma_{ab}^A \gamma^{Bbc} + \gamma_{ab}^B \gamma^{A bc} = 2\delta^{AB} \delta_a^c, \quad \gamma_{ij}^{A'} \gamma^{B'jk} + \gamma_{ij}^{B'} \gamma^{A'jk} = 2\delta^{A'B'} \delta_i^k. \quad (4.6)$$

Using (2.3), (2.10) and (4.6) it is straightforward to check that the objects (4.5) are real under usual complex conjugation and obey standard properties of $SO(3)$ matrices,

$$\begin{aligned} U_{B'}^{A'} U_{B'}^{C'} &= \delta^{A'C'}, & \varepsilon^{A'B'C'} U_{A'}^1 U_{B'}^2 U_{C'}^3 &= 1, & \overline{U_{A'}^{B'}} &= U_{A'}^{B'}, \\ V_B^A V_B^C &= \delta^{AC}, & \varepsilon^{ABC} V_A^1 V_B^2 V_C^3 &= 1, & \overline{V_B^A} &= V_B^A. \end{aligned} \quad (4.7)$$

In terms of the SO(3)-harmonics (4.5) the bosonic components of the superfield (3.32) can be written as

$$\begin{aligned}
W = & \varphi^A V_A^1 - i\phi^{A'} U_{A'}^1 + \frac{1}{\sqrt{2}}(\theta_\alpha^1 \theta_\beta^4 \sigma^{m\alpha}{}_{\dot{\alpha}} \sigma^{n\beta\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}}^3 \bar{\theta}_{\dot{\beta}}^2 \sigma^{m\dot{\alpha}}{}_{\alpha} \sigma^{n\alpha\dot{\beta}}) F_{mn} \\
& + 2\theta^{1\alpha} \bar{\theta}^{2\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^A (V_A^2 + iV_A^3) + 2\theta^{4\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^A (V_A^2 - iV_A^3) \\
& - 2i\theta^{4\alpha} \bar{\theta}^{2\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi^{A'} (U_{A'}^2 - iU_{A'}^3) - 2i\theta^{1\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi^{A'} (U_{A'}^2 + iU_{A'}^3) \\
& - 4\theta^{1\alpha} \theta^{4\beta} \bar{\theta}^{3\dot{\alpha}} \bar{\theta}^{2\dot{\beta}} \partial_{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}} (V_A^1 \varphi^A + iU_{A'}^1 \phi^{A'}), \tag{4.8}
\end{aligned}$$

where we have defined the SO(3) triplets of the scalars as

$$\varphi^A = \frac{1}{2} \gamma_{ab}^A \varphi^{ab}, \quad \phi^{A'} = \frac{1}{2} \gamma_{ij}^{A'} \phi^{ij}. \tag{4.9}$$

In what follows, we will use the expression (4.8) to analyze the component structure of the action (4.1) in the bosonic sector.

4.3 The F^4/X^4 term

To identify the F^4/X^4 term in the effective action, we neglect terms with derivatives of scalar fields in (4.8), so that

$$W = \varphi^A V_A^1 - i\phi^{A'} U_{A'}^1 + \frac{1}{\sqrt{2}}(\theta_\alpha^1 \theta_\beta^4 \sigma^{m\alpha}{}_{\dot{\alpha}} \sigma^{n\beta\dot{\alpha}} + \bar{\theta}_{\dot{\alpha}}^3 \bar{\theta}_{\dot{\beta}}^2 \sigma^{m\dot{\alpha}}{}_{\alpha} \sigma^{n\alpha\dot{\beta}}) F_{mn}. \tag{4.10}$$

Substituting (4.10) into (4.1) and integrating over the Grassmann variables by the rule (4.2), we find

$$\Gamma_{F^4} = \frac{1}{4} \int d^4x dU dV H^{(4)}(\varphi^A V_A^1 - i\phi^{A'} U_{A'}^1) \left[F_{mn} F^{nk} F_{kl} F^{lm} - \frac{1}{4} (F_{pq} F^{pq})^2 \right]. \tag{4.11}$$

Here we have applied the standard identity for the trace of four sigma-matrices,

$$\text{tr } \tilde{\sigma}^m \sigma^n \tilde{\sigma}^p \sigma^q = -2i\epsilon^{mnpq} + 2(\eta^{mn}\eta^{pq} + \eta^{np}\eta^{mq} - \eta^{mp}\eta^{nq}). \tag{4.12}$$

Choosing now the function H as in (4.4), we expand it in the Taylor series over $i\phi^{A'} U_{A'}^1$,

$$\begin{aligned}
H^{(4)}(\varphi^A V_A^1 - i\phi^{A'} U_{A'}^1) &= \sum_{n=0}^{\infty} \frac{1}{n!} H^{(n+4)}(\varphi^A V_A^1) (-i\phi^{A'} U_{A'}^1)^n \\
&= c \sum_{n=0}^{\infty} \frac{(-1)^{n+1} (n+3)!}{n!} \frac{(-i\phi^{A'} U_{A'}^1)^n}{(\varphi^A V_A^1)^{n+4}}. \tag{4.13}
\end{aligned}$$

Here $H^{(n)}$ stands for the n 'th derivative of the function H with respect to its argument. Substituting this decomposition into (4.11) and computing the harmonic integral over dU using (B.3), we obtain

$$\begin{aligned}
\Gamma_{F^4} &= -\frac{c}{4} \int d^4x dV [F_{mn} F^{nk} F_{kl} F^{lm} - \frac{1}{4} (F_{pq} F^{pq})^2] \sum_{n=0}^{\infty} (2n+2)(2n+3) \frac{(-\phi^{A'} \phi^{A'})^n}{(\varphi^A V_A^1)^{2n+4}} \\
&= \frac{c}{2} \int d^4x dV [F_{mn} F^{nk} F_{kl} F^{lm} - \frac{1}{4} (F_{pq} F^{pq})^2] \frac{\phi^{A'} \phi^{A'} - 3(\varphi^A V_A^1)^2}{[\phi^{D'} \phi^{D'} + (\varphi^D V_D^1)^2]^3}. \tag{4.14}
\end{aligned}$$

It is interesting to note that the series in the first line in (4.14) reduced to the concise analytical expression given in the second line. This allows us to expand the expression in the second line in (4.14) in a series over another argument, $\varphi^A V_A^1$, and compute the harmonic integral over dV ,

$$\begin{aligned}\Gamma_{F^4} &= \frac{c}{2} \int d^4x dV \frac{F_{mn} F^{nk} F_{kl} F^{lm} - \frac{1}{4} (F_{pq} F^{pq})^2}{(\phi^{D'} \phi^{D'})^2} \sum_{n=0}^{\infty} (-1)^n (2n+1)(n+1) \frac{(\varphi^A V_A^1)^{2n}}{(\phi^{A'} \phi^{A'})^n} \\ &= \frac{c}{2} \int d^4x \frac{F_{mn} F^{nk} F_{kl} F^{lm} - \frac{1}{4} (F_{pq} F^{pq})^2}{(\phi^{D'} \phi^{D'})^2} \sum_{n=0}^{\infty} (-1)^n (n+1) \left(\frac{\varphi^A \varphi^A}{\phi^{A'} \phi^{A'}} \right)^n.\end{aligned}\quad (4.15)$$

This series can be easily summed up, and we find the following result

$$\Gamma_{F^4} = \frac{c}{2} \int d^4x \frac{F_{mn} F^{nk} F_{kl} F^{lm} - \frac{1}{4} (F_{pq} F^{pq})^2}{(\phi^{A'} \phi^{A'} + \varphi^A \varphi^A)^2}.\quad (4.16)$$

Note that the scalar fields in the denominator appear in an $\text{SO}(6)$ -invariant form.

4.4 The WZ term

To identify the Wess-Zumino term in the effective action, we omit the Maxwell field strength from the expansion in (4.8), so that now

$$\begin{aligned}W &= V_A^1 \varphi^A - i U_{A'}^1 \phi^{A'} \\ &\quad + 2\theta^{1\alpha} \bar{\theta}^{2\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^A (V_A^2 + i V_A^3) + 2\theta^{4\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \varphi^A (V_A^2 - i V_A^3) \\ &\quad - 2i\theta^{4\alpha} \bar{\theta}^{2\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi^{A'} (U_{A'}^2 - i U_{A'}^3) - 2i\theta^{1\alpha} \bar{\theta}^{3\dot{\alpha}} \partial_{\alpha\dot{\alpha}} \phi^{A'} (U_{A'}^2 + i U_{A'}^3) \\ &\quad - 4\theta^{1\alpha} \theta^{4\beta} \bar{\theta}^{3\dot{\alpha}} \bar{\theta}^{2\dot{\beta}} \partial_{\alpha\dot{\alpha}} \partial_{\beta\dot{\beta}} (V_A^1 \varphi^A + i U_{A'}^1 \phi^{A'}).\end{aligned}\quad (4.17)$$

The terms in the last line do not contribute to the WZ term, as they contain two space-time derivatives acting on the same scalar. Substituting the remaining terms into (4.1) and computing the integral over the Grassmann variables, we find

$$\begin{aligned}\Gamma_{\text{WZ}} &= \int d^4x dU dV H^{(4)} (V_D^1 \varphi^D - i U_{D'}^1 \phi^{D'}) \partial_{\alpha\dot{\alpha}} \varphi^A \partial_{\beta\dot{\beta}} \varphi^B \partial^{\beta\dot{\alpha}} \phi^{A'} \partial^{\alpha\dot{\beta}} \phi^{B'} \\ &\quad \times (V_A^2 + i V_A^3) (V_B^2 - i V_B^3) (U_{A'}^2 - i U_{A'}^3) (U_{B'}^2 + i U_{B'}^3)\end{aligned}\quad (4.18)$$

With $\partial_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^m \partial_m$, we apply the trace formula for the sigma-matrices (4.12) and single out only the term with the antisymmetric ε -tensor,

$$\Gamma_{\text{WZ}} = -8i\varepsilon^{mnpq} \int d^4x dU dV H^{(4)} (V_D^1 \varphi^D - i U_{D'}^1 \phi^{D'}) \partial_m \varphi^A \partial_n \varphi^B \partial_p \phi^{A'} \partial_q \phi^{B'} V_A^2 V_B^3 U_{A'}^2 U_{B'}^3.\quad (4.19)$$

Substituting the series decomposition (4.13) into (4.19) and computing the integral over the U -harmonics by the rule (B.4), we obtain

$$\begin{aligned}\Gamma_{\text{WZ}} &= -16c \varepsilon^{mnpq} \varepsilon_{A'B'C'} \int d^4x dV \sum_{n=0}^{\infty} (n+1)(n+2)(-1)^n \frac{(\phi^{D'} \phi^{D'})^n}{(V_D^1 \varphi^D)^{2n+5}} \\ &\quad \times \phi^{A'} \partial_p \phi^{B'} \partial_q \phi^{C'} \partial_m \varphi^A \partial_n \varphi^B V_A^2 V_B^3 \\ &= -32c \varepsilon^{mnpq} \varepsilon_{A'B'C'} \int d^4x dV \frac{V_C^1 \varphi^C}{[\phi^{D'} \phi^{D'} + (V_D^1 \varphi^D)^2]^3} \phi^{A'} \partial_p \phi^{B'} \partial_q \phi^{C'} \partial_m \varphi^A \partial_n \varphi^B V_A^2 V_B^3.\end{aligned}\quad (4.20)$$

(As in (4.14), the series allowed explicit resummation.) Next, we expand the integrand in a series over $V_D^1 \varphi^D$ and perform the integration over the V -harmonics in a similar way,

$$\begin{aligned} \Gamma_{\text{WZ}} = & -8c \varepsilon^{mnpq} \int d^4x \frac{1}{(\phi^{D'} \phi^{D'})^3} \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)}{2n+3} \left(\frac{\varphi^D \varphi^D}{\phi^{D'} \phi^{D'}} \right)^n \\ & \times (\varepsilon_{A'B'C'} \phi^{A'} \partial_p \phi^{B'} \partial_q \phi^{C'}) (\varepsilon_{ABC} \varphi^A \partial_m \varphi^B \partial_n \varphi^C). \end{aligned} \quad (4.21)$$

The series can be summed up, and we obtain the following result

$$\Gamma_{\text{WZ}} = -2c \varepsilon^{mnpq} \int d^4x \frac{f(z)}{(\phi^{D'} \phi^{D'})^3} (\varepsilon_{A'B'C'} \phi^{A'} \partial_p \phi^{B'} \partial_q \phi^{C'}) (\varepsilon_{ABC} \varphi^A \partial_m \varphi^B \partial_n \varphi^C), \quad (4.22)$$

where

$$f(z) = \frac{z^2 - 1}{z^2(z^2 + 1)^2} + \frac{\arctan z}{z^3}, \quad z^2 = \frac{\varphi^A \varphi^A}{\phi^{A'} \phi^{A'}}. \quad (4.23)$$

Let us now introduce the normalized scalars,

$$Y^A = \frac{\varphi^A}{\sqrt{\varphi^B \varphi^B + \phi^{B'} \phi^{B'}}}, \quad Y^{A'} = \frac{\phi^{A'}}{\sqrt{\varphi^B \varphi^B + \phi^{B'} \phi^{B'}}}, \quad (4.24)$$

which lie on the unit five-sphere, $Y^A Y^A + Y^{A'} Y^{A'} = 1$. In terms of these scalars, the action (4.22) reads

$$\Gamma_{\text{WZ}} = -2c \varepsilon^{mnpq} \int d^4x g(y) (\varepsilon_{ABC} Y^A \partial_p Y^B \partial_q Y^C) (\varepsilon_{A'B'C'} Y^{A'} \partial_m Y^{B'} \partial_n Y^{C'}), \quad (4.25)$$

where

$$g(y) = \frac{y^4 - 1}{y^2} + \frac{(y^2 + 1)^3}{y^3} \arctan y, \quad y^2 = \frac{Y^A Y^A}{Y^{A'} Y^{A'}}. \quad (4.26)$$

Comparing (4.25) with (1.1), we see that we have perfect agreement provided

$$c = \frac{1}{32\pi^2}. \quad (4.27)$$

The F^4/X^4 term (4.16) then also has the coefficient as given in [1]. According to the analysis presented in [1] (see also references quoted there), this is the minimal value of the constant c allowed by the topological quantization condition. It corresponds to the case when the SYM gauge group $\text{SU}(2)$ is broken to $\text{U}(1)$.

4.5 $\text{SU}(N)$ gauge group

The effective action (4.1) with the function $H(W)$ given in (4.4) can be easily generalized to describe the $\mathcal{N} = 4$ SYM effective action on the Coulomb branch in the case when the gauge group $\text{SU}(N)$ is spontaneously broken to its maximal abelian subgroup $[\text{U}(1)]^{N-1}$. The superfield W in this case is a traceless diagonal $N \times N$ matrix in the Cartan subalgebra of $\mathfrak{su}(N)$,

$$W = \text{diag}(W^1, W^1, \dots, W^N), \quad \sum_{i=1}^N W^i = 0, \quad (4.28)$$

with all eigenvalues being distinct: $W^i \neq W^j$ if $i \neq j$. The effective action in this case reads

$$\Gamma = \frac{1}{32\pi^2} \int d\zeta du dv \sum_{i < j}^N \ln \frac{|W^i - W^j|}{\Lambda}. \quad (4.29)$$

For the case of the gauge group $SU(2)$ spontaneously broken down to $U(1)$ this sum reduces to (4.1) with (4.4). For non-unitary gauge groups, the effective action can be written in a similar way with the summation over the positive roots of the gauge algebra (see, e.g., [11, 15, 25, 26, 27] for similar generalizations in the $\mathcal{N} = 2$ superspace).

5 Conclusions

In this paper, we developed a formulation of the $\mathcal{N} = 4$ SYM effective action in a novel $\mathcal{N} = 4$ $d = 4$ bi-harmonic superspace. The coordinates of this superspace involve two copies of the standard $SU(2)$ harmonic variables, u_i^\pm and v_a^\pm , which allowed us to make manifest the $SU(2) \times SU(2) \sim SO(3) \times SO(3)$ subgroup of the full $SU(4) \sim SO(6)$ R-symmetry group of the $\mathcal{N} = 4$ SYM theory.

The idea of introducing this superspace was inspired by our previous work [1], where we showed that the effective Lagrangian in the $\mathcal{N} = 4$ SYM theory written in the four-dimensional form can be made manifestly invariant under one of the following three subgroups of the full $SO(6)$ R-symmetry: $SO(5)$, $SO(4) \times SO(2)$ or $SO(3) \times SO(3)$. In [1], we explored two superspace formulations of the $\mathcal{N} = 4$ SYM effective action which correspond to the $SO(5)$ and $SO(4) \times SO(2)$ subgroups, whereas in the present paper we constructed a novel superspace which makes the $SO(3) \times SO(3)$ subgroup manifest. We therefore demonstrated that for each of the maximal non-anomalous subgroups of the $SU(4)$ R-symmetry there exists the corresponding superspace description of the $\mathcal{N} = 4$ SYM effective action.

Representing the $SU(4)$ index I by a pair of $SU(2)$ indices (i, a) , we found that the antisymmetric $\mathcal{N} = 4$ superfield strength W^{IJ} is equivalently described by two *symmetric* $SU(2)$ tensors, $W^{(ij)}$ and $W^{(ab)}$. We found constraints for these superfields which restrict their field content to that of the on-shell $\mathcal{N} = 4$ supergauge multiplet and which are equivalent to the standard constraints for the superfield W^{IJ} . At this stage, the constraints still mix all the superfield components, so that one needs both $W^{(ij)}$ and $W^{(ab)}$ to describe the $\mathcal{N} = 4$ SYM multiplet.

As the next step, we contracted $W^{(ij)}$ and $W^{(ab)}$ with the harmonic variables and obtained six $\mathcal{N} = 4$ superfields W , W' , $W^{(\pm 2, 0)}$, $W^{(0, \pm 2)}$ with definite charges under the $U(1) \times U(1)$ subgroup of $SU(2) \times SU(2)$. We then found that we could *decouple* the corresponding constraints, so that any one of these six bi-harmonic superfields can be used to describe the $\mathcal{N} = 4$ SYM multiplet. The advantage of using these superfields instead of W^{IJ} is that part of the constraints can be solved by simply passing to the corresponding analytic subspaces. We explicitly solved the remaining constraints for the superfields W and $W^{(2, 0)}$, and gave the corresponding component field decompositions. We showed that all the auxiliary field components vanish, whereas the physical components correspond to the on-shell $\mathcal{N} = 4$ gauge multiplet.

We argued that the chargeless harmonic superfield W is particularly convenient for describing the low-energy effective $\mathcal{N} = 4$ SYM action. We explicitly demonstrated this

by presenting the leading four-derivative part of the effective action in the bi-harmonic superspace. Scale invariance required this part to be just $\ln W$ integrated over the corresponding analytic subspace. The similarity of this result with the corresponding expression in a totally different $\mathcal{N} = 4$ USp(4) harmonic superspace, see [1], is quite striking. Even more striking is the way the algebra worked out in our proof that the expected F^4/X^4 and WZ terms are indeed contained in this superfield expression.

It is the main (and somewhat unexpected) result of the present paper that the $\mathcal{N} = 4$ SYM low-energy effective action becomes extremely simple when considered in the $\mathcal{N} = 4$ bi-harmonics superspace.

It still remains to be understood whether and how this effective action could be described in the $\mathcal{N} = 4$ SU(4) harmonic superspace [28, 29, 30, 31] or in the $\mathcal{N} = 3$ SU(3) harmonic superspace [32, 33]. In [1], we pointed out that these are *not* particularly convenient superspaces because SU(4) and SU(3) have ‘mild’ anomalies that, via the properties of the WZ term, prevent these symmetries to be manifestly realized. (They are still symmetries of the effective action, but must transform the Lagrangian into a total divergence.) However, the corresponding descriptions could still exist.

An important application of our results would be to try to construct higher-derivative parts of the effective action in the available $\mathcal{N} = 4$ superspaces introduced in [6, 1] and in the present paper. It would also be interesting to see if the on-shell restrictions could be relaxed in these superspaces. Then one could try to rederive our results through explicit perturbative quantum calculations in the off-shell harmonic superspaces (cf. [34]). We leave these problems for future studies.

Acknowledgments

The research of D.B. was supported by the Department of Energy Grant No. DE-FG02-97ER41029. The work of I.S. was supported by the Marie Curie research fellowship No. 236231, “QuantumSupersymmetry”; by RFBR grants Nr. 09-02-00078 and 11-02-90445; by grant for LRSS, project No. 3558.2010.2.

A Covariant derivatives in analytic coordinates

In this appendix, we give the explicit form of the covariant spinor and harmonic derivatives in the analytic subspaces defined in (2.16).

A.1 Spinor and harmonic derivatives in the subspace A_1

Covariant spinor derivatives:

$$\begin{aligned}
D_\alpha^2 &= -\frac{\partial}{\partial\theta^{2\alpha}}, & D_\alpha^1 &= \frac{\partial}{\partial\theta^{1\alpha}} + 2i\bar{\theta}^{1\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\
D_\alpha^3 &= -\frac{\partial}{\partial\theta^{3\alpha}}, & D_\alpha^4 &= \frac{\partial}{\partial\theta^{4\alpha}} + 2i\bar{\theta}^{4\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\
\bar{D}_{\dot{\alpha}}^1 &= -\frac{\partial}{\partial\bar{\theta}^{1\dot{\alpha}}}, & \bar{D}_{\dot{\alpha}}^2 &= \frac{\partial}{\partial\bar{\theta}^{2\dot{\alpha}}} - 2i\theta^{2\alpha}\partial_{\alpha\dot{\alpha}}, \\
\bar{D}_{\dot{\alpha}}^4 &= -\frac{\partial}{\partial\bar{\theta}^{4\dot{\alpha}}}, & \bar{D}_{\dot{\alpha}}^3 &= \frac{\partial}{\partial\bar{\theta}^{3\dot{\alpha}}} - 2i\theta^{3\alpha}\partial_{\alpha\dot{\alpha}}.
\end{aligned} \tag{A.1}$$

Covariant harmonic derivatives:

$$\begin{aligned}
D_{A_1}^{(2,0)} &= D^{(2,0)} + 2i\theta^{2\alpha}\bar{\theta}^{4\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + 2i\theta^{1\alpha}\bar{\theta}^{3\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta_\alpha^1\frac{\partial}{\partial\theta_\alpha^3} + \theta_\alpha^2\frac{\partial}{\partial\theta_\alpha^4} + \bar{\theta}_{\dot{\alpha}}^4\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^2} + \bar{\theta}_{\dot{\alpha}}^3\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^1}, \\
D_{A_1}^{(-2,0)} &= D^{(-2,0)} + 2i\theta^{4\alpha}\bar{\theta}^{2\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + 2i\theta^{3\alpha}\bar{\theta}^{1\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta_\alpha^3\frac{\partial}{\partial\theta_\alpha^1} + \theta_\alpha^4\frac{\partial}{\partial\theta_\alpha^2} + \bar{\theta}_{\dot{\alpha}}^2\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^4} + \bar{\theta}_{\dot{\alpha}}^1\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^3}, \\
D_{A_1}^{(0,2)} &= D^{(0,2)} + 2i\theta^{1\alpha}\bar{\theta}^{2\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + 2i\theta^{3\alpha}\bar{\theta}^{4\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta_\alpha^1\frac{\partial}{\partial\theta_\alpha^2} + \theta_\alpha^3\frac{\partial}{\partial\theta_\alpha^4} + \bar{\theta}_{\dot{\alpha}}^4\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^3} + \bar{\theta}_{\dot{\alpha}}^2\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^1}, \\
D_{A_1}^{(0,-2)} &= D^{(0,-2)} + 2i\theta^{4\alpha}\bar{\theta}^{3\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + 2i\theta^{2\alpha}\bar{\theta}^{1\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta_\alpha^2\frac{\partial}{\partial\theta_\alpha^1} + \theta_\alpha^4\frac{\partial}{\partial\theta_\alpha^3} + \bar{\theta}_{\dot{\alpha}}^3\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^4} + \bar{\theta}_{\dot{\alpha}}^1\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^2}.
\end{aligned} \tag{A.2}$$

In the above expressions, $\partial_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x_{A_1}^m}$.

A.2 Spinor and harmonic derivatives in the subspace A_2

Covariant spinor derivatives:

$$\begin{aligned}
\bar{D}_{\dot{\alpha}}^1 &= -\frac{\partial}{\partial\bar{\theta}^{1\dot{\alpha}}}, & \bar{D}_{\dot{\alpha}}^3 &= +\frac{\partial}{\partial\bar{\theta}^{3\dot{\alpha}}} - 2i\theta^{3\alpha}\partial_{\alpha\dot{\alpha}}, \\
\bar{D}_{\dot{\alpha}}^2 &= +\frac{\partial}{\partial\bar{\theta}^{2\dot{\alpha}}}, & \bar{D}_{\dot{\alpha}}^4 &= -\frac{\partial}{\partial\bar{\theta}^{4\dot{\alpha}}} - 2i\theta^{4\alpha}\partial_{\alpha\dot{\alpha}}, \\
D_\alpha^3 &= -\frac{\partial}{\partial\theta^{3\alpha}}, & D_\alpha^1 &= +\frac{\partial}{\partial\theta^{1\alpha}} + 2i\bar{\theta}^{1\dot{\alpha}}\partial_{\alpha\dot{\alpha}}, \\
D_\alpha^4 &= +\frac{\partial}{\partial\theta^{4\alpha}}, & D_\alpha^2 &= -\frac{\partial}{\partial\theta^{2\alpha}} + 2i\bar{\theta}^{2\dot{\alpha}}\partial_{\alpha\dot{\alpha}}.
\end{aligned} \tag{A.3}$$

Covariant harmonic derivatives:

$$\begin{aligned}
D_{A_2}^{(2,0)} &= D^{(2,0)} - 2i\theta^{2\alpha}\bar{\theta}^{4\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + 2i\theta^{1\alpha}\bar{\theta}^{3\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta_\alpha^1\frac{\partial}{\partial\theta_\alpha^3} + \theta_\alpha^2\frac{\partial}{\partial\theta_\alpha^4} + \bar{\theta}_{\dot{\alpha}}^4\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^2} + \bar{\theta}_{\dot{\alpha}}^3\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^1}, \\
D_{A_2}^{(-2,0)} &= D^{(-2,0)} - 2i\theta^{4\alpha}\bar{\theta}^{2\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + 2i\theta^{3\alpha}\bar{\theta}^{1\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta_\alpha^3\frac{\partial}{\partial\theta_\alpha^1} + \theta_\alpha^4\frac{\partial}{\partial\theta_\alpha^2} + \bar{\theta}_{\dot{\alpha}}^2\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^4} + \bar{\theta}_{\dot{\alpha}}^1\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^3}, \\
D_{A_2}^{(0,2)} &= D^{(0,2)} + \theta_\alpha^1\frac{\partial}{\partial\theta_\alpha^2} + \theta_\alpha^3\frac{\partial}{\partial\theta_\alpha^4} + \bar{\theta}_{\dot{\alpha}}^4\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^3} + \bar{\theta}_{\dot{\alpha}}^2\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^1}, \\
D_{A_2}^{(0,-2)} &= D^{(0,-2)} + \theta_\alpha^2\frac{\partial}{\partial\theta_\alpha^1} + \theta_\alpha^4\frac{\partial}{\partial\theta_\alpha^3} + \bar{\theta}_{\dot{\alpha}}^3\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^4} + \bar{\theta}_{\dot{\alpha}}^1\frac{\partial}{\partial\bar{\theta}_{\dot{\alpha}}^2}.
\end{aligned} \tag{A.4}$$

In the above expressions, $\partial_{\alpha\dot{\alpha}} = \sigma_{\alpha\dot{\alpha}}^m \frac{\partial}{\partial x_{A_2}^m}$.

B SO(3) harmonic integrals

The SO(3) harmonic variables U and V are nothing but the usual SO(3) matrices with the properties (4.7). The relations (4.5) among these harmonics and the SU(2) ones, together with the SU(2) harmonic integration rules (4.3), yield

$$\begin{aligned} \int dU \, 1 &= 1, & \int dU (\text{non-singlet SO(3) irreducible representation}) &= 0, \\ \int dV \, 1 &= 1, & \int dV (\text{non-singlet SO(3) irreducible representation}) &= 0. \end{aligned} \quad (\text{B.1})$$

There are two obvious consequences of these SO(3) harmonic integration rules,

$$\int dV \, V_A^1 V_B^1 = \frac{1}{3} \delta_{AB}, \quad \int dV \, V_A^1 V_B^2 V_C^3 = \frac{1}{3!} \varepsilon_{ABC}, \quad (\text{B.2})$$

where ε_{ABC} is the totally antisymmetric SO(3) tensor. After a bit of combinatorics, we obtain the following generalization of these two harmonic integrals

$$\int dV \, V_{A_1}^1 \dots V_{A_k}^1 = \begin{cases} \frac{1}{k+1} \delta_{(A_1 A_2 \dots A_{k-1} A_k)}, & k = 2n \\ 0 & k = 2n + 1, \end{cases} \quad (\text{B.3})$$

$$\int dV \, V_{A_1}^1 \dots V_{A_k}^1 V_A^1 V_B^2 V_C^3 = \begin{cases} \frac{\delta_{(A_1 A_2 \dots A_{k-1} A_k \varepsilon A) BC}}{2(k+3)}, & k = 2n \\ 0 & k = 2n + 1. \end{cases} \quad (\text{B.4})$$

The same identities hold also for the integrals with the U -harmonics.

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